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On the Flow of Electric Current in Semi-Infinite Media in Which the Specific Resistance is a Function of the Depth

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VIII. *On the Flow of Electric Current in Semi-Infinite Media in which the Specific Resistance is a Function of the Depth.*

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Section 1. *Introduction.*

In a previous paper* the writer has dealt with the flow of electric current from a point-electrode at the surface flowing into a stratified medium each layer of which is of constant specific resistance. In the present paper a general method is developed for determining the distribution of surface potential when the specific resistance is a continuous function of the depth. When this problem has been solved, the surface-potential distribution for any arrangement of discrete or continuously distributed electrodes, which it may be convenient to use in geophysical prospecting by this method, may be obtained by addition or by integration if continuous line-electrodes are employed.

In dealing with the point-electrode there is some advantage in working with the electrical current-function ψ. Using cylindrical co-ordinates (r, z) referred to the electrode as origin, the radial and axial components of current-flow across unit area are given by

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial z} = -\frac{1}{\rho}\frac{\partial V}{\partial r} \quad \text{and} \quad u_z = -\frac{1}{r}\frac{\partial \psi}{\partial r} = -\frac{1}{\rho}\frac{\partial V}{\partial z}, \quad . \quad . \quad . \quad . \quad (1)$$

while the lines of flow are given by $\psi = const.$

On eliminating the potential V, it is readily seen that ψ satisfies the differential equation

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho\,\frac{\partial \psi}{\partial z}\right) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where ρ is the specific resistance of the medium at the point (r, z).

If ψ_0 be the value of ψ at any point on the axis of z, and ψ_s that on the surface $z = 0$, it is easily shown that the total current I introduced at the electrode is given by

$$I = 2\pi\,(\psi_0 - \psi_s). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

* King, 'Proc. Roy. Soc.' A, vol. 139, p. 237 (1933). The reader is also referred to a recent paper by Muskat ('Physics,' vol. 4, p. 129 (1933)).

In geophysical prospecting, it is convenient to plot the quantity $\bar{\rho}$, called the "surface-gradient characteristic," against the distance r from the electrode. This observable quantity is given by

$$\bar{\rho} = -\frac{2\pi r^2}{I}\frac{\delta V_s}{\delta r} = \frac{2\pi\rho_s}{I} r \left[\frac{\partial\psi}{\partial z}\right]_{z=0}, \quad \ldots \ldots \ldots \quad (4)$$

where $\delta V_s/\delta r$ is the surface-potential gradient, and ρ_s the specific resistance at the surface $z = 0$, supposed to be known.

For a homogeneous medium, it is evident that $\bar{\rho}/\rho_s = 1$. When $\bar{\rho}/\rho_s$ is plotted against the distance from the electrode we obtain a straight line through the point $\bar{\rho}/\rho_s = 1$ parallel to the r axis. When the specific resistance varies with the depth continuously or discontinuously, we obtain a curve departing from this straight line, from which the nature of the "electrical depth-constants" is to be inferred. As yet no general method of uniquely determining these "depth constants" from an analysis of the "surface-gradient characteristic" appears to have been devised, although the solution of the problem would seem to be facilitated by a process of analysis of field observations based on HANKEL'S inversion theorem.*

As it is of practical importance in geophysical prospecting to be able to estimate the effect of specific resistance gradients in the determination of electrical depth constants, we proceed to outline the general theory for determining the surface-potential gradient for a given law of variation of the resistivity with depth. This requires us to develop the theory of Sturm–Liouville expansions pertinent to the problem under consideration, as applied to a medium of continuously variable specific resistance bounded at $z = h$ by a perfectly insulating or conducting plane. Then by making $h \to \infty$, a method is suggested for obtaining a solution of the problem for an infinitely deep medium. It is also possible to extend the theory to a discontinuously stratified medium in each layer of which the specific resistance varies continuously with the depth. This later development must, however, be left for a future communication.

Section 2. Determination of ψ for Medium of Varying Conductivity Bounded by Insulating Plane at $z = h$.

It is easily seen that a general solution of the differential equation (2) is given by the series

$$\psi = \sum_s A_s Z(\lambda_s, z) \,.\, r\lambda_s K_1(r\lambda_s) + C\int_0^z \frac{dz}{\rho}, \quad \ldots \ldots \ldots \quad (5)$$

where $K_1(x)$, in the standard notation, is the Bessel function of imaginary argument having the convenient property

$$\lim_{x\to 0} xK_1(x) = \lim_{r\to 0} r\lambda K_1(\nu\lambda) = 1. \quad \ldots \ldots \ldots \quad (6)$$

* See 'Proc. Roy. Soc.,' A, vol. 139, p. 237 (1933).

$Z(\lambda_s, z)$ is a solution of the differential equation of the Sturm-Liouville type*

$$\frac{d^2Z}{dz^2} + \frac{1}{\rho}\frac{d\rho}{dz}\frac{dZ}{dz} + \lambda^2 Z = 0, \quad \ldots \ldots \quad (7)$$

so chosen that $Z(\lambda_s, z) = 0$ over the plane $z = 0$.

The summation is extended over successive values of the parameter λ obtained as roots of an *equation of condition.*

Since $Z(\lambda_s, 0) = 0$, it follows from (5) that $\psi_s = 0$ on $z = 0$. Since there is no flow across the insulating plane $z = h$, it follows from (3) that $I = 2\pi(\psi)_{z=h}$ for all values of r, as a result of which it is seen from (5) that

$$Z(\lambda_s, h) = 0, \quad \text{and} \quad I = 2\pi C \int_0^h dz/\rho. \quad \ldots \ldots \quad (8)$$

Thus the *equation of condition* determining the parameters λ_s is

$$Z(\lambda_s, h) = 0. \quad \ldots \ldots \quad (9)$$

If we now take a point on the axis of z where $r = 0$, it follows that $I = 2\pi\psi_0$, so that, making use of (6), the coefficients A_s are determined by the Sturm–Liouville expansion

$$-C\int_h^z \frac{dz}{\rho} = \sum_s A_s Z(\lambda_s, z), \qquad (0 < z < h), \quad \ldots \ldots \quad (10)$$

where C is given in terms of I by (8).

If Z_1 and Z_2 be any two solutions of (7) corresponding to any two different values λ_1 and λ_2 of the parameter λ, it is easy to prove, by multiplying the differential equation in Z_1 by Z_2, and that in Z_2 by Z_1, substracting and integrating between limits z_1 and z_2, that

$$({\lambda_2}^2 - {\lambda^2}_1)\int_{z_1}^{z_2} \rho Z_1 Z_2 \, dz = \left[\rho\left\{Z_2\frac{dZ_1}{dz} - Z_1\frac{dZ_2}{dz}\right\}\right]_{z_1}^{z_2}. \quad \ldots \ldots \quad (11)$$

It is convenient to regard the left-hand side of (10) as a particular solution of equation (7) corresponding to $\lambda = 0$. Denoting $Z_0 = \int_h^z dz/\rho$, it follows as a particular case of (11) that

$$\lambda^2\int_{z_1}^{z_2} \rho Z_0 Z \, dz = \left[\rho\left\{Z\frac{dZ_0}{dz} - Z_0\frac{dZ}{dz}\right\}\right]_{z_1}^{z_2}, \qquad (\lambda \neq 0) \ldots \ldots \quad (12)$$

If now we take the limits z_1 and z_2 to be 0 and h respectively, then in view of the

* INCE, "Ordinary Differential Equations," Chaps. X, XI, Longmans, Green and Co., 1927. Differential equations of the type (7) first made their appearance in the theory of heat-flow in a bar whose thermal conductivity is not constant, but is a function of the distance from one end. The theory of the present paper is of course applicable to problems of heat flow, and to distribution of potential in dielectrics of variable specific inductive capacity.

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equation of condition (9), which requires that Z_1 and Z_2 both vanish at $z = 0$ and $z = h$, we have,

$$\int_0^h \rho Z_1 Z_2 \, dz = 0 , \quad (\lambda_1 \neq \lambda_2) . \qquad (13)$$

We may now obtain the coefficients A_s appearing in (10) by multiplying each side by $\rho Z(\lambda_s, z)$ and integrating between the limits 0 and h. It follows from (13) that all terms on the right vanish except those for which $\lambda = \lambda_s$. On making use of (12), the integral on the left is easily evaluated. We thus obtain the result

$$\frac{I}{2\pi}\left(\frac{dZ_s}{dz}\right)_{z=0} = A_s \int_0^h \frac{\rho}{\rho_s}\left(\frac{dZ_s}{dz}\right)^2 dz = A_s \lambda_s^2 \int_0^h \frac{\rho}{\rho_s} Z_s^2 \, dz, \qquad (14)$$

where we have written Z_s to denote $Z(\lambda_s, z)$.

Here ρ_s denotes the specific resistance at the surface $z = 0$. The last equality is derived from the second by integrating by parts and making use of (7), which may be written in the form

$$\frac{d}{dz}\left(\rho \frac{dZ}{dz}\right) + \lambda^2 Z \rho = 0. \qquad (15)$$

Since equation (11) is valid for any two values λ_1 and λ_2 of the parameter λ, it follows that

$$\int^z \rho Z^2 \, dz = \lim_{\lambda_2 \to \lambda_1} \left[\frac{\rho}{\lambda_2^2 - \lambda_1^2}\left\{Z_2 \frac{dZ_1}{dz} - Z_1 \frac{dZ_2}{dz}\right\}\right].$$

Evaluating the limit in the usual way, we obtain the result

$$\int^z \rho Z^2 \, dz = \tfrac{1}{2}\frac{\rho}{\lambda}\left\{\frac{\partial Z}{\partial \lambda} \cdot \frac{\partial Z}{dz} - Z \frac{\partial^2 Z}{\partial \lambda \, \partial z}\right\}, \qquad (16)*$$

which can usually be further simplified when λ is a root of an equation of condition.

Special case.—In many problems we may write $Z(\lambda, z) = Z\{\lambda(z + a)\}$, where a is a constant.

We then have

$$\frac{\partial Z}{\partial \lambda} = \frac{z}{\lambda}\frac{dZ}{dz}, \qquad \frac{\partial^2 Z}{\partial \lambda \, \partial z} = \frac{1}{\lambda}\frac{dZ}{dz} + \frac{z}{\lambda}\frac{d^2 Z}{dz^2},$$

and the general result (16) takes the more explicit form

$$\int^z \rho\,[Z\{\lambda(z + a)\}]^2 \, dz = \frac{\rho(z + a)}{2\lambda^2}\left\{\left(\frac{dZ}{dz}\right)^2 + \lambda^2 Z^2 + \frac{1}{\rho}\frac{d\rho}{dz} \cdot Z \frac{dZ}{dz}\right\} - \frac{\rho}{2\lambda^2} Z \frac{dZ}{dz}, \qquad (17)$$

in the derivation of which we have used equations (12) and (15).

* This procedure is sometimes referred to as an application of l'Hospital's rule. (WATSON, "Bessel Functions," § 5.11.)

If we now make use of this result in (14), we obtain the following formula for the coefficient A_s, noting that Z vanishes for $z = 0$ and $z = h$.

$$A_s = \frac{I\rho_s}{\pi} \frac{(dZ_s/dz)_{z=0}}{\Big[\rho\,(z+a)\,\{(dZ_s/dz)^2 + \lambda_s^2 Z_s^2\}\Big]_{z=0}^{z=h}} \quad . \quad . \quad . \quad (18)$$

The term $\lambda_s^2 Z_s^2$ is retained in the formula, although $Z_s \to 0$ at both limits, in view of the requirements of Section 4 in which the limiting form of A_s when $h \to \infty$ is useful in attaining ψ for a medium of infinite depth.

It should be emphasized that (18) is only applicable when the law of variation of ρ with depth is of such a character that the differential equation (7) for Z admits of solutions in terms of a variable of the form $\lambda\,(z + a)$. In other cases the more general integral (16) should be employed.

When the coefficients A_s have been determined by the use of the appropriate formulæ, it follows from equations (4), (5) and (8) that the "surface gradient characteristic" is given by the expansion

$$\frac{\bar{\rho}}{\rho_s} = \frac{2\pi}{I}\, r \left(\frac{\partial \psi}{dz}\right)_{z=0} = \frac{r}{\rho_s \int_0^h dz/\rho} + \frac{2\pi}{I} \sum_s A_s\,(dZ_s/dz)_{z=0}\,.\,r^2 \lambda_s K_1\,(r\lambda_s), \quad . \quad . \quad (19)$$

the summation being extended over values of λ_s given by the equation of condition $Z\,(\lambda_s, h) = 0$.

Section 3. *Determination of ψ for Medium of Varying Conductivity Bounded by a Perfectly Conducting Plane at $z = h$.*

Since the stream-lines in this case cut the plane $z = h$ everywhere at right angles, the boundary condition is evidently $(\partial\psi/dz)_{z=h}$ for all values of r. It follows from (5) that $C = 0$, provided that ρ is finite at the boundary, while the equation of condition determining the parameters λ_s in the expansion

$$\psi = \sum_s A_s Z\,(\lambda_s, z)\,.\,r\lambda_s K_1\,(r\lambda_s) \quad . \quad . \quad . \quad (20)$$

is $(dZ/dz)_{z=h} = 0$. By taking a point on the axis $r = 0$ in the interval $0 < z < h$, when $I = 2\pi\psi_0$, we see that the coefficients A_s are determined from the expansion

$$I/(2\pi) = \sum_s A_s Z\,(\lambda_s, z). \quad . \quad . \quad . \quad (21)$$

Proceeding as in Section 2, making use of (13) and (15), we find that the general formulæ for A_s are the same as (14). An inspection of (17) shows, furthermore, that the special formula (18) with $(dZ_s/dz)_{z=h} = 0$ is applicable, while the first term in the expansion (19) is omitted, in view of the fact that $C = 0$.

Section 4. *Determination of* ψ *for Medium of Varying Conductivity of Infinite Depth.*

It is obvious that the physical problem limits the character of ρ as a function of the depth. The fundamental equation (7) requires that both ρ and $d\rho/dz$ be continuous functions of z and single valued in the interval for which the equation is valid. At the surface $z = 0$ the specific resistance ρ must be neither zero nor infinite. If the law of variation of specific resistance with depth gives zero or infinite values of ρ in the interval $0 < z < \infty$, the problem reduces to that of Sections 2 or 3, h being the least value of z for which ρ is infinite or zero respectively.

When ρ is neither zero nor infinite in the interval $0 < z < \infty$, we seek a solution for ψ of the form

$$\psi = \int_0^\infty \phi(\lambda)\, Z(\lambda, z)\, r\lambda K_1(r\lambda)\, d\lambda + C \int_0^z \frac{dz}{\rho}, \qquad (22)$$

where $Z(\lambda, z)$ is a solution of (7) which vanishes for $z = 0$.

The total current flow across a surface of revolution swept out by a plane curve having one extremity on the axis $r = 0$ at $z = \infty$, and the other on the surface $z = 0$ at $r = \infty$ is I. It follows from (3) that the constant C is given by

$$I = 2\pi C \int_0^\infty \frac{dz}{\rho}, \qquad (23)$$

if the law of variation of specific resistance is such that the above integral is finite: otherwise the last term in (22) is not required.

Since I is also the total current flow across a surface of revolution swept out by a plane curve having its extremities on the axis $r = 0$ at any depth z, and the other on the surface $z = 0$ at any distance r from the axis, it follows, since $Z(\lambda, 0) = 0$, that $\phi(\lambda)$ must be determined from the integral equation

$$\frac{I}{2\pi} \frac{\int_z^\infty \frac{dz}{\rho}}{\int_0^\infty \frac{dz}{\rho}} = \int_0^\infty \phi(\lambda)\, Z(\lambda, z)\, d\lambda. \qquad (24)$$

When $C = 0$, the left-hand side of the above equation is simply $I/(2\pi)$.

For a limited number of special laws of variation of $\rho(z)$ with depth z, the differential equation (7) for $Z(\lambda, z)$ gives rise to functions for which known inversion formulæ exist.

When this is not so, a solution of the problem may sometimes be derived by introducing a perfectly insulating or conducting boundary at $z = h$. The solution of this "bounded problem" may be obtained by the procedure of Section 2 or 3.

If we now make $h \to \infty$ the roots of the equation of condition will become everywhere dense, and the series (5) for ψ will, in the limit, become a definite integral of the form (22).

The following simple example will illustrate the procedure.

Consider a homogeneous medium bounded by an insulating plane at $z = h$. The solution is

$$\psi = \frac{I}{2\pi}\left[\frac{z}{h} + \frac{2r}{h}\sum_{s=1}^{\infty} \sin \lambda_s z K_1 (r\lambda_s)\right]. \quad \ldots \ldots \quad (25)^*$$

The summation is extended over the positive roots of the equation of condition $\sin \lambda_s h = 0$, so that $\lambda_s = s\pi/h$.

Since $\lambda_{s+1} - \lambda_s = \pi/h$, we may write (25) in the form

$$\psi = \frac{I}{2\pi^2}\left[\frac{\pi z}{h} + 2r\sum_{s=1}^{\infty} \sin \lambda_s z K_1 (r\lambda_s)(\lambda_{s+1} - \lambda_s)\right]. \quad \ldots \quad (26)$$

If we now make $h \to \infty$, the first term drops out, and in the limit the summation becomes a definite integral having limits 0 and ∞. We thus have for an infinite medium of constant specific resistance

$$\psi = \frac{Ir}{\pi^2}\int_0^{\infty} K_1 (\lambda r) \sin \lambda z \, d\lambda = \frac{I}{2\pi}\frac{z}{(r^2 + z^2)^{\frac{1}{2}}}. \quad \ldots \ldots \quad (27)$$

The same result follows if we apply this procedure to the solution for a homogeneous medium bounded by a perfectly conducting plane at $z = h$. A further example is given in § 7 of the writer's previous paper.

A generalization of this procedure is now obvious. Referring to equation (5) we write the summation in the form

$$\sum_s A_s Z(\lambda_s, z) \, . \, r\lambda_s K_1 (r\lambda_s) = \sum_s \frac{A_s}{\lambda_{s+1} - \lambda_s} \, . \, Z(\lambda_s, z) \, . \, r\lambda_s K_1 (r\lambda_s)(\lambda_{s+1} - \lambda_s), \quad \ldots \quad (28)$$

where λ_{s+1} and λ_s are two successive roots of the equation of condition for which a general formula may sometimes be derived.†

If we then examine the ratio $A_s/(\lambda_{s+1} - \lambda_s)$, with A_s, given by (18), for large values of h, using the equation of condition, it is found in a number of particular cases that in the limit as $h \to \infty$ and the distribution of roots becomes everywhere dense,

$$\lim \{A_s/(\lambda_{s+1} - \lambda_s)\} = \phi(\lambda_s),$$

a known function of λ_s. In these circumstances the right-hand side of (28) becomes, in the limit,

$$\int \phi(\lambda) Z(\lambda, z) \, . \, r\lambda K_1 (r\lambda) \, d\lambda.$$

* 'Proc. Roy. Soc.,' A, vol. 139, p. 245 (1933).

† The general theory of the distribution of roots of functions of the Sturm–Liouville type is dealt with by INCE, "Ordinary Differential Equations," Chap. XXI, Longmans, Green and Co., London, 1927. Formulæ connected with the roots of Bessel functions are given in GRAY, MATTHEWS and MACROBERT, "Bessel Functions," Macmillan's, London, p. 260, 1931; WATSON, "Theory of Bessel Functions," Chap. X, 1922, Camb. Univ. Press.

The limits of integration are usually 0 to ∞, but in general will depend on the distribution of the roots of the equation of condition.

When the medium is bounded by a perfectly insulating or conducting plate at $z = h$, the methods of Sections 2 and 3 may be applied to determine ψ for a more general type of vertical line-electrode from which the current flow at depth z, between z and $z + dz$ is $\mathrm{I}(z)\,dz$, and $\mathrm{I}(z)$ is a single-valued arbitrary function continuous or discontinuous. For a medium of infinite depth we have to solve an integral equation of the type (24) in which the left-hand is a prescribed function of z. The procedure of the present section enables $\phi(\lambda)$ to be found, and suggests for functions $\mathrm{Z}(\lambda, z)$, solutions of the Sturm–Liouville equation (7), types of inversion formulæ of which the FOURIER integral, HANKEL's inversion formula and WEBER's integral theorem are particular examples.

Although the procedure of making h tend to infinity may introduce serious theoretical difficulties, the writer has found the method useful in suggesting the form of the required solution which it is then possible to establish rigorously by contour integration.

Section 5. *Special Solutions for* ψ *by the Use of Known Inversion Formulæ.*

Although the methods outlined in the preceding sections can be depended on to give solutions for any law of variation of specific resistance with depth for which the differential equation (7) for $\mathrm{Z}(\lambda, z)$ entering into the general formula (27) can be solved, it is often advisable to attempt a solution which depends on some known or independently derived inversion formula. We proceed to consider a number of simple illustrations.

Example 1. *Semi-Infinite Medium of Constant Specific Resistance,* $\rho/\rho_s = 1$.

The general procedure is well illustrated by this trivial example. An assumed solution of the form

$$\psi = \int_0^\infty \phi(\lambda) \sin \lambda z \,.\, \lambda r\, \mathrm{K}_1(\lambda r)\, d\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

evidently satisfies the differential equation (2), (or 7), and gives $\psi_s = 0$ over the surface $z = 0$.

In view of (3), $\psi = \psi_0$ when $r \to 0$, so that for all values of z between the limits $0 < z < \infty$, we must have

$$\frac{\mathrm{I}}{2\pi} = \int_0^\infty \phi(\lambda) \sin \lambda z\, d\lambda, \qquad (0 < z < \infty). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

On making use of the known result,

$$\int_0^\infty \sin \lambda z \,.\, \frac{d\lambda}{\lambda} = \tfrac{1}{2}\pi,$$

valid for $0 < z < \infty$, the solution of (30) as an integral equation is, obviously, $\phi(\lambda) = \mathrm{I}/(\pi^2 \lambda)$.

If we now substitute in (29), the solution for ψ is

$$\psi = \frac{I}{\pi^2}\int_0^\infty \lambda r\, K_1(\lambda r) \sin \lambda z \frac{d\lambda}{\lambda} = \frac{I}{2\pi}\frac{z}{(r^2+z^2)^{\frac{1}{2}}}, \quad (31)$$

while an application of (4) to either term of (31) gives $\rho/\rho_s = 1$, as we should expect.*

Example 2. *Exponential Resistance Gradient*, $\rho = \rho_s\, e^{-2mz}$, $(0 < m < \infty)$.

Equation (7) for Z takes the simple form

$$\frac{d^2 Z}{dz^2} - 2m\frac{dZ}{dz} + \lambda^2 Z = 0. \quad (32)$$

The solution of this equation which vanishes at $z = 0$ is

$$Z(\lambda, z) = e^{mz}\sin \mu z, \qquad \text{where} \quad \mu = (\lambda^2 - m^2)^{\frac{1}{2}}. \quad (33)$$

We therefore assume a solution for ψ of the form

$$\psi = e^{mz}\int_0^\infty \phi(\mu)\sin \mu z \,.\, r\lambda K_1(r\lambda)\, d\mu + C\int_0 \frac{dz}{\rho}, \quad (34)$$

which is of the type (25), except that it is more convenient to use the variable μ.

Since the flow across an infinitely deep cylinder at $r \to \infty$ remains finite, it is readily seen that we must have $C = 0$.

Considering the flow across a surface of revolution swept out by a plane curve between any point on the axis of z and the surface, we have, according to (3), since $\psi_s = 0$, $2\pi\psi_0 = I$ for all values of z. On writing $r \to 0$ in (34), we see that the solution of the problem depends on the possibility of determining $\phi(\mu)$ from the integral equation

$$\frac{I}{2\pi}e^{-mz} = \int_0^\infty \phi(\mu)\sin \mu z\, d\mu. \quad (35)$$

If we now made use of the well-known integral

$$\int_0^\infty \frac{\mu \sin \mu z}{\mu^2 + m^2} d\mu = \tfrac{1}{2}\pi\, e^{-mz}, \quad (m > 0, z > 0),$$

we evidently have

$$\phi(\mu) = \frac{I}{\pi^2}\frac{\mu}{\mu^2 + m^2}. \quad (36)$$

It is readily seen that differentiation of (34) under the integral sign is permissible, the simplest criterion for differentiability being satisfied. Equation (4) then gives

$$\frac{\bar{\rho}}{\rho_s} = \frac{2\pi}{I} r \left(\frac{\partial \psi}{\partial z}\right)_{z=0} = \frac{2}{\pi} r^2 \int_0^\infty \frac{\mu^2}{(\mu^2+m^2)^{\frac{1}{2}}} K_1\{r(\mu^2+m^2)^{\frac{1}{2}}\}\, d\mu. \quad (37)$$

* In deriving (31) we have integrated by parts and made use of well-known formulæ involving integrals of the K-functions, for which the reader is referred to WATSON's treatise, § 13.21 (10). The notation of WATSON's treatise is employed throughout this paper.

Making use of the known integral

$$\int_0^\infty \frac{K_\nu \{a(t^2+z^2)^{\frac{1}{2}}\}}{(t^2+z^2)^{\frac{1}{2}\nu}} t^{2\mu+1}\, dt = \frac{2^\mu \Gamma(\mu+1)}{a^{\mu+1} z^{\nu-\mu-1}} K_{\nu-\mu-1}(az), \quad \ldots \quad (38)^*$$

valid for $a > 0$ and $R(\mu) > -1$, and remembering that $K_{-\frac{1}{2}}(t) = K_{\frac{1}{2}}(t) = (\frac{1}{2}\pi/t)^{\frac{1}{2}} e^{-t}$, we find

$$\bar{\rho}/\rho_s = e^{-mr}, \quad \ldots \quad (39)$$

a remarkably simple result.

Example 3. *Exponential Resistance Gradient*, $\rho = \rho_s e^{2mz}$ $(0 < m < \infty)$.

Following the procedure of Example 2, the appropriate form for ψ is

$$\psi = e^{-mz} \int_0^\infty \phi(\mu) \sin \mu z \,.\, r\lambda K_1(r\lambda)\, d\mu + C \int_0^z dz/\rho. \quad \ldots \quad (40)$$

The last term introduces the term $\{C/(2m\rho_s)\}(1 - e^{-2mz})$. By considering the flow of current across a cylinder of infinite radius and depth, we readily find $C/(2m\rho_s) = I/(2\pi)$. We now make use of equation (3), $2\pi(\psi_0 - \psi_s) = I$; for the determination of $\phi(\mu)$ we thus have the equation

$$I/(2\pi) = e^{-mz} \int_0^\infty \phi(\mu) \sin \mu z\, d\mu + \{I/(2\pi)\}(1 - e^{-2mz}),$$

which is identical with (35), so that making use of (36) we have

$$\psi = \frac{I}{\pi^2} e^{-mz} \int_0^\infty \frac{\mu \sin \mu z}{\mu^2 + m^2} r\lambda\, K_1(r\lambda)\, d\mu + \frac{I}{2\pi}(1 - e^{-2mz}), \quad \ldots \quad (41)$$

in which $\lambda^2 = \mu^2 + m^2$.

We readily find as before, making use of (4) and the integral (38), that in this case

$$\bar{\rho}/\rho_s = 2mr + e^{-mr}. \quad \ldots \quad (42)$$

Example 4. *Parabolic Resistance Gradient*, $\rho/\rho_s = (1 + z/a)^2$, $(a > 0)$.

It is easily proved that the differential equation (7) for $Z(\lambda, z)$ may be solved in the form $Z = v\sqrt{\rho_s/\rho}$, provided that v satisfies the equation

$$\frac{d^2 v}{dz^2} + \left(\lambda^2 - \tfrac{1}{2}\frac{dP}{dz} - \tfrac{1}{4}P^2\right) v = 0, \qquad \text{where} \quad P = \frac{1}{\rho}\frac{d\rho}{dz}. \quad \ldots \quad (43)$$

This result enables us to state a number of resistance gradients for which ψ may be

* WATSON, § 13.47 (6).

completely determined. In the present problem the equation for v gives, simply, $v = \sin \lambda z$, and by assuming a value for ψ of the form

$$\psi = \int_0^\infty \phi(\lambda)\, Z(\lambda, z) \,.\, r\lambda\, K_1(r\lambda)\, d\lambda + C \int_0^z dz/\rho, \quad \ldots \ldots \quad (44)$$

it is a simple matter to prove that

$$\psi = -\frac{I}{2\pi} \frac{a}{a+z} \left\{1 - \frac{z}{(r^2+z^2)^{\frac{1}{2}}}\right\}, \quad \ldots \ldots \quad (45)$$

and

$$\bar{\rho}/\rho_s = 1 + r/a, \quad \ldots \ldots \quad (46)$$

while the potential V at any point, (r, z), is found to be

$$V = \frac{I\rho_s}{2\pi} \left\{\frac{a+z}{(r^2+z^2)^{\frac{1}{2}}} - \sinh^{-1}\left(\frac{z}{r}\right) - \log\left(\frac{r}{a}\right)\right\}. \quad \ldots \ldots \quad (47)$$

Example 5. *Resistance Gradient*, $\rho/\rho_s = \operatorname{sech}^2 ma \,.\, \cosh^2 m(z-a)$, $(0 \leq a < \infty)$.

It is evident from (43) that the differential equation for $Z(\lambda, z)$ may be solved in terms of elementary functions if P satisfies an equation of the type $\frac{1}{2}\frac{dP}{dz} + \frac{1}{4}P^2 - m^2 = 0$, since we then have $P = 2m \tanh m(z-a)$. Hence by (43) we have the above law for the resistance gradient, a being a constant of integration.

The equation for v gives $v = \sin \{z(\lambda^2 - m^2)^{\frac{1}{2}}\}$, so that, writing $\mu^2 = \lambda^2 - m^2$, we assume a solution of the form

$$\psi = \operatorname{sech} m(z-a) \,.\, \int_0^\infty \phi(\mu) \sin \mu z \,.\, r\lambda\, K_1(r\lambda)\, d\mu + C \tanh m(z-a), \quad (48)$$

and readily find $\phi(\mu) = (I/\pi^2) \cosh ma \,.\, \mu/(\mu^2 + m^2)$, and ultimately

$$\bar{\rho}/\rho_s = mr\, e^{-ma} \operatorname{sech} ma + e^{-mr}. \quad \ldots \ldots \quad (49)$$

When mz is small, we have, $\rho/\rho_s \sim 1 - 2mz \tanh ma + \ldots$, while when mr is small, $\bar{\rho}/\rho_s \sim 1 - mr \tanh ma + \ldots$.

The resistance gradient just described is of some geophysical interest as ρ/ρ_s decreases to a minimum at $z = a$, and then increases indefinitely. The curve for $\bar{\rho}/\rho_s$ plotted against r shows a corresponding minimum at $e^{mr} = e^{ma} \cosh ma$, and then increases, ultimately following an asymptote through the origin of slope $me^{-ma} \operatorname{sech} ma$.*

* Field observations of $\bar{\rho}/\rho_s$ sometimes exhibit a general trend of this character which the result of this example shows is not necessarily associated with the existence of strata involving discontinuities of specific resistance.

Section 6. *Resistance Gradients*, $\rho/\rho_s = (1 + z/a)^m$, $(0 < a < \infty, -\infty < m < -\infty)$.

In order that ρ be neither zero nor infinite in the interval $0 < z < \infty$, the constant a must be positive. Resistance gradients for which a is negative require to be dealt with by the methods of Section 2 or 3. The constant m is unrestricted, and may be positive or negative.

In this instance the differential equation for Z (λ, z) is

$$\frac{d^2Z}{dz^2} + \frac{m}{a+z}\frac{dZ}{dz} + \lambda^2 Z = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

of which the general solution is known to be

$$Z\{\lambda(a+z)\} = (a+z)^{-\nu}[AJ_\nu\{\lambda(a+z)\} + BJ_{-\nu}\{\lambda(a+z)\}], \quad . \quad . \quad (51)^*$$

where for brevity we write $\nu = \frac{1}{2}(m-1)$.

It is convenient to express $J_{-\nu}$ in terms of J_ν and Y_ν by the formula

$$Y_\nu(x) = \{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)\}\operatorname{cosec}\nu\pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)^\dagger$$

Case (i).—$1 < m < \infty$, or $0 < \nu < \infty$.—In the notation adopted, the solution of (50), which vanishes for $z = 0$, may be written

$$Z\{\lambda(a+z)\} = (a+z)^{-\nu}[Y_\nu\{\lambda(a+z)\}J_\nu(a\lambda) - J_\nu\{\lambda(a+z)\}Y_\nu(a\lambda)]. \quad (53)$$

We have, furthermore,

$$\int_0^z \frac{dz}{\rho} = \frac{a}{2\nu\rho_s}\left\{1 - \left(\frac{a}{a+z}\right)^{2\nu}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

The appropriate expression for ψ is

$$\psi = \int_0^\infty \phi(\lambda)\, Z\{\lambda(a+z)\} \,.\, r\lambda K_1(r\lambda)\, d\lambda + C\int_0^z \frac{dz}{\rho}. \quad . \quad . \quad . \quad . \quad (55)$$

provided that $\phi(\lambda)$ can be determined to satisfy the requirements of the problem.

If we consider the flow of current across an infinitely deep cylinder of infinite radius we find from (54) that provided $0 < \nu < \infty$,

$$C = 2\nu\rho_s I/(2\pi a). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

We now use (3), which expresses the fact that the flow of current across a surface of revolution bounded by the plane $z = 0$, and having its vertex at any point of the z-axis

* WATSON, § 4.31.

† WATSON, § 3.61 (3).

is equal to I for all values of z; as a consequence of making $r \to 0$ in (55), and using (54) and (56), we find that the integral equation for the determination of $\phi(\lambda)$ is

$$\frac{\mathrm{I}}{2\pi}\frac{a^{2\nu}}{(a+z)^{\nu}} = \int_0^\infty \phi(\lambda)\,[\mathrm{Y}_\nu\{\lambda(a+z)\}\,\mathrm{J}_\nu(a\lambda) - \mathrm{J}_\nu\{\lambda(a+z)\}\,\mathrm{Y}_\nu(a\lambda)]\,d\lambda. \quad (57)^*$$

We are led from the considerations of Section 4 to expect that a solution of the above equation can be found. In fact, it can be shown as a particular case of a general contour integral that, for unrestricted values of ν, positive or negative, $(z > 0)$,

$$\int_0^\infty \frac{\mathrm{Y}_\nu\{\lambda(a+z)\}\,\mathrm{J}_\nu(a\lambda) - \mathrm{J}_\nu\{\lambda(a+z)\}\,\mathrm{Y}_\nu(a\lambda)}{\mathrm{Y}_\nu^2(a\lambda)+\mathrm{J}_\nu^2(a\lambda)}\frac{d\lambda}{\lambda} = \frac{\pi}{2}\left(\frac{a}{a+z}\right)^{|\nu|}. \quad (58)\dagger$$

On comparing (57) and (58), we see that $\phi(\lambda)$ is at once determined, and equation (1) gives

$$\mathrm{V} = -\frac{\mathrm{I}\rho_s}{2\pi}\left[\left(1+\frac{z}{a}\right)^{\nu+1}\frac{2}{\pi}\int_0^\infty \frac{\mathrm{Y}_{\nu+1}\{\lambda(a+z)\}\,\mathrm{J}_\nu(a\lambda) - \mathrm{J}_{\nu+1}\{\lambda(a+z)\}\,\mathrm{Y}_\nu(a\lambda)}{\mathrm{Y}_\nu^2(a\lambda)+\mathrm{J}_\nu^2(a\lambda)}\,\mathrm{K}_0(r\lambda)\,d\lambda + \frac{2\nu}{a}\log r\right]. \quad (59)$$

On making use of the relation

$$\mathrm{Y}_{\nu+1}(x)\,\mathrm{J}_\nu(x) - \mathrm{J}_{\nu+1}(x)\,\mathrm{Y}_\nu(x) = -2/(\pi x), \quad . \; . \; . \; . \; . \; . \quad (60)\ddagger$$

we easily find that $(d\mathrm{Z}/dz)_{z=0} = (2/\pi)\,a^{-(\nu+1)}$, and ultimately deduce after a few simple reductions

$$\frac{\bar{\rho}}{\rho_s} = \frac{2\pi}{\mathrm{I}}\,r\left(\frac{\partial\psi}{dz}\right)_{z=0} = \frac{r}{a}\left\{2\nu + \frac{4}{\pi^2}\int_0^\infty \frac{r\mathrm{K}_1(r\lambda)\,d\lambda}{\mathrm{Y}_\nu^2(a\lambda)+\mathrm{J}_\nu^2(a\lambda)}\right\}, \quad . \; . \; . \; . \quad (61)$$

where $\nu = \frac{1}{2}(m-1)$, and $0 \leqq \nu < \infty$.

By using the procedure of Section 4, the reader will have no difficulty in establishing the formula

$$\psi = \frac{\mathrm{I}}{\pi^2}\int_0^\infty \frac{a^\nu \mathrm{Z}\{\lambda(a+z)\}\,.\,r\mathrm{K}_1(r\lambda)\,d\lambda}{\mathrm{Y}_\nu^2(a\lambda)+\mathrm{J}_\nu^2(a\lambda)} + \frac{\mathrm{I}\nu}{\pi a}\int_0^z \frac{\rho_s}{\rho}\,dz, \quad . \; . \; . \; . \quad (62)$$

valid for positive values of ν, as the limiting form of the expansion (5) when h is made to tend to infinity. From this result, (61) follows immediately from (4).

When ν is negative, the appropriate solution of (50) is

$$\mathrm{Z}\{\lambda(a+z)\} = (a+z)^{-\nu}\,[\mathrm{Y}_{-\nu}\{\lambda(a+z)\}\,\mathrm{J}_{-\nu}(a\lambda) - \mathrm{J}_{-\nu}\{\lambda(a+z)\}\,\mathrm{Y}_{-\nu}(a\lambda)]. \quad (63)$$

* [*Added Feb.* 12, 1934.—The determination of $\phi(\lambda)$ in (57) may also be effected by the use of WEBER's integral theorem. ('Math. Ann.,' vol. 6, p. 154 (1873).) See also a paper by TITCHMARSH, 'Proc. London M th. Soc.,' vol. 22, p. 15 (1924).]

† The proof of (58) as a result of contour integration is outlined in Appendix I. The integral is discontinuous at $z = 0$.

‡ WATSON, § 3.63 (12).

Since the last term of (55) drops out when ν is negative, we have to determine $\phi(\lambda)$ in the expression

$$\psi = \int_0^\infty \phi(\lambda)\, Z_\nu\{\lambda(a+z)\} \,.\, r\lambda K_1(r\lambda)\, d\lambda, \quad \ldots \ldots \quad (64)$$

from a solution of

$$\frac{I}{2\pi}(a+z)^{-n}\int_0^\infty \phi(\lambda)\,[Y_n\{\lambda(a+z)\}\, J_n(a\lambda) - J_n\{\lambda(a+z)\}\, Y_n(a\lambda)]\, d\lambda, \quad (65)$$

in which we have written $\nu = -n$, so that n is positive.

We may now use (58) to determine $\phi(\lambda)$, and, as it is at once evident from (52) that $Y_\nu^2 + J_\nu^2 = Y_{-\nu}^2 + J_{-\nu}^2$, we finally obtain the formula

$$\frac{\bar{\rho}}{\rho_s} = \frac{4}{\pi^2}\frac{r}{a}\int_0^\infty \frac{rK_1(r\lambda)\, d\lambda}{Y_\nu^2(a\lambda) + J_\nu^2(a\lambda)} \qquad (-\infty < \nu \leqq 0), \quad \ldots \ldots \quad (66)$$

when $\nu = \frac{1}{2}(m-1)$.

The expression for the potential V is of the same form as (59) with the omission of the term in log r.

Section 7. *Discussion of Particular Cases of Resistance Gradient* $\rho/\rho_s = (1 + z/a)^m$.

The formulæ (61) and (66) are of considerable generality, and the integral converges rapidly, lending itself readily to graphical computation. In a few cases, when m is an even integer, the integral may be evaluated in terms of known functions of r/a, as in the following examples.

Example 1. $\rho/\rho_s = (1 + z/a)^{-2}$, $\nu = -\frac{3}{2}$.

Since $Y_{-\frac{3}{2}}^2(\lambda a) + J_{-\frac{3}{2}}^2(\lambda a) = \left(\frac{2}{\pi\lambda a}\right)\{1 + (\lambda a)^{-2}\}$, we have

$$\int_0^\infty \frac{r\, K_1(r\lambda)\, d\lambda}{Y_{-\frac{3}{2}}^2(\lambda a) + J_{-\frac{3}{2}}^2(\lambda a)} = \frac{\pi}{2} ra\left[\int_0^\infty \lambda\, K_1(r\lambda)\, d\lambda - \int_0^\infty \frac{\lambda\, K_1(r\lambda)\, d\lambda}{1+\lambda^2 a^2}\right]$$
$$= \tfrac{1}{2}\pi\, ra\,[\tfrac{1}{2}\pi\, r^{-2} - \tfrac{1}{4}\pi^2\{Y_{-1}(r/a) - H_{-1}(r/a)\}] \,.\; . \quad (67)^*$$

and from the known formulæ

$$Y_{-1}(x) = -Y_1(x), \quad \text{and} \quad H_{-1}(x) = 1/\pi - H_1(x), \quad \ldots \ldots \quad (68)^\dagger$$

for Bessel functions and Struve functions we finally obtain the integrated form of (66) in terms of tabulated functions

$$\bar{\rho}/\rho_s = 1 + \tfrac{1}{2}r^2/a^2 + \tfrac{1}{2}\pi\,(r^2/a^2)\,\{Y_1(r/a) - H_1(r/a)\}. \quad \ldots \ldots \quad (69)$$

* WATSON, § 13.52 (9). The properties of the Struve functions $H_\nu(x)$ are given in WATSON's treatise, § 10.4, while extensive tables are given following Chap. XX.

† WATSON, § 3.51 (4) and § 3.54 (2): also § 10.4 (5).

From the known approximations for $Y_1(r/a)$ and $H_1(r/a)$, valid for small values of r/a, it follows that

$$\bar{\rho}/\rho_s \sim 1 - r/a + (r^2/a^2)(\gamma - \log 2 + \tfrac{2}{3}) + \tfrac{1}{2}(r^3/a^3)\log(r/a) + \dots .$$

Example 2. *Homogeneous Medium*, $\rho/\rho_s = 1$, $\nu = -\frac{1}{2}$.

We have here $Y_{-\frac{1}{2}}^2(\lambda a) + J_{-\frac{1}{2}}^2(\lambda a) = 2/(\pi\lambda a)$, and (66) gives, as we should expect,

$$\frac{\bar{\rho}}{\rho_s} = \frac{2}{\pi}\frac{r}{a}\int_0^\infty a\lambda K_1(r\lambda)\,d\lambda = 1.$$

Example 3. *Linear Gradient*, $\rho/\rho_s = 1 + z/a$, $\nu = 0$.

In this case both (61) and (66) agree in giving,

$$\frac{\bar{\rho}}{\rho_s} = \frac{4}{\pi^2}\frac{r}{a}\int_0^\infty \frac{rK_1(r\lambda)\,d\lambda}{Y_0^2(\lambda a) + J_0^2(\lambda a)}, \qquad (70)$$

which, apparently, cannot be evaluated in terms of known functions.

Example 4. *Parabolic Gradient*, $\rho/\rho_s = (1 + z/a)^2$, $\nu = \frac{1}{2}$.

In this case (61) gives the simple law

$$\bar{\rho}/\rho_s = 1 + r/a, \qquad (71)$$

a result already independently obtained in Section 5.

Example 5. $\rho/\rho_s = (1 + z/a)^4$, $\nu = \frac{3}{2}$.

Proceeding as in Example 1, but employing the appropriate formula (61), we find

$$\bar{\rho}/\rho_s = 1 + 3r/a + \tfrac{1}{2}r^2/a^2 + \tfrac{1}{2}\pi(r^2/a^2)\{Y_1(r/a) - H_1(r/a)\}. \qquad (72)$$

For small values of r/a, this formula gives

$$\rho/\rho_s \sim 1 + 2(r/a) + \dots .$$

Example 6. $\bar{\rho}/\rho_s = (1 + z/a)^6$, $\nu = \frac{5}{2}$.

We easily find,

$$Y_{\frac{5}{2}}^2(\lambda a) + J_{\frac{5}{2}}^2(\lambda a) = (2/\pi)(\lambda^4a^4 + 3\lambda^2a^2 + 9)/(\lambda a)^5.$$

Since the factors of the quadratic in λ^2a^2 are complex, it is readily seen by resolution into partial fractions that the infinite integral in (61) can be made to depend on an integral of the type

$$\int_0^\infty \frac{xK_1(ax)}{x^2+k^2}\,dx = \frac{\pi^2}{4}\{Y_{-1}(ak) - H_{-1}(ak)\},$$

dealt with in (72), but as k is complex for half odd integral values of $\nu \geqq \frac{5}{2}$, the final result depends on Y_1- and H_1-functions of complex argument. The corresponding general formula for $Y_\nu^2 + J_\nu^2$ when ν is half an odd integer yields a polynomial in odd powers of $(\lambda a)^{-1}$ with simple coefficients.*

General Remarks.—It is interesting to note that when $a \to \infty$, $\rho/\rho_s \to 1$ for all values of m. From the asymptotic formulæ for Y_ν and J_ν, it follows that

$$Y_\nu^2(\lambda a) + J_\nu^2(\lambda a) \sim (2/\pi)/(\lambda a),$$

and both (61) and (66) yield the correct result, $\bar{\rho}/\rho_s = 1$.

When the values of ν become large, whether integral or not, we may advantageously employ the approximate formulæ for functions of large order.†

In particular, if we make use of the results (ν large),

$$\left.\begin{aligned} &Y_\nu^2(\nu \sec \beta) + J_\nu^2(\nu \sec \beta) \sim \cot \beta(/\tfrac{1}{2}\pi\nu), \quad (0 < \beta < \tfrac{1}{2}\pi) \\ &Y_\nu^2(\nu \operatorname{sech} \alpha) + J_\nu^2(\nu \operatorname{sech} \alpha) \sim e^{-2\nu(\tanh\alpha - \alpha)} \coth \alpha \end{aligned}\right\}, \quad . \quad . \quad . \quad (73)$$

it is a simple matter to prove that the formulæ of Section 6 are valid for the limiting case of the gradient

$$\rho/\rho_s = \lim_{m\to\infty} \left(1 + \frac{2\mu}{m} z\right)^m = e^{2\mu z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

and finally to reproduce the results of Section 5, examples 2 and 3.

Section 8. *Direct Determination of the Potential* V.

When the stream function ψ has been determined by the methods described in the preceding sections, the potential V may be determined by effecting the integrations implied in equation (1).

In some cases there is an advantage in determining V directly, the solutions thus obtained being expressed in terms of the Bessel functions J_0 and J_1.

It follows from (1) that the differential equation for V is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{\rho}\frac{\partial V}{dr}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial V}{\partial z}\right) = 0\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

in terms of V the radial and axial current components are given by

$$u_r = -\frac{1}{\rho}\frac{\partial V}{\partial r} \quad \text{and} \quad u_z = -\frac{1}{\rho}\frac{\partial V}{\partial z}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

In the present paper, in which we deal with semi-infinite media for which ρ is

* WATSON, § 7.51 (5).

† WATSON, § 8.22 and § 8.4.

continuous and neither zero nor infinite in the interval $0 \leqq z < \infty$, we confine our attention to possible solutions of the type

$$V = \int_0^\infty \phi(\lambda) \, Z(\lambda, z) \, . \, J_0(\lambda r) \, d\lambda. \quad \ldots \ldots \ldots \quad (77)$$

It is easily seen from (75) that $Z(\lambda, z)$ satisfies the differential equation

$$\frac{d^2Z}{dz^2} - \frac{1}{\rho}\frac{d\rho}{dz}\frac{dZ}{dz} - \lambda^2 Z = 0. \quad \ldots \ldots \ldots \quad (78)$$

In forming solutions of the type (77), the solution (if any) which vanishes as $z \to \infty$ must be taken.

If, now, we integrate the current flow over a cylinder of radius r and depth z, the result must be I for all values of r and z. For the particular type of solution (77), this leads to an integral equation for $\phi(\lambda)$

$$-\frac{I}{2\pi} = \int_0^\infty \frac{\phi(\lambda)}{\lambda^2} \cdot \frac{1}{\rho_s}\left(\frac{dZ}{dz}\right)_{z=0} . \, r\lambda J_1(r\lambda) \, d\lambda. \quad \ldots \ldots \ldots \quad (79)$$

From Hankel's inversion theorem it immediately follows that the solution of (84) is

$$\frac{\phi(\lambda)}{\lambda}\left(\frac{dZ}{dz}\right)_{z=0} = -\frac{I\rho_s}{2\pi}. \quad \ldots \ldots \ldots \quad (80)$$

With this value of $\phi(\lambda)$, the potential is given by (77), and, in particular, writing $z = 0$, we have the surface potential V_s, so that the "surface-gradient characteristic" is given by

$$\frac{\bar{\rho}}{\rho_s} = -\frac{2\pi r^2}{I\rho_s}\frac{dV_s}{dr}. \quad \ldots \ldots \ldots \quad (81)$$

In these problems, the divergent integral

$$V_0 = \int_0^\infty J_0(\lambda r) \, d\lambda/\lambda$$

sometimes appears in the solution for V. Since

$$\frac{dV_0}{dr} = -\int_0^\infty J_1(\lambda r) \, d\lambda = -1/r,$$

the integral V_0 is to be interpreted as $-\log r + const.$, and its appearance in the expression for V implies the existence of an asymptote through the origin in the graph of $\bar{\rho}/\rho_s$ plotted against r.

We proceed to illustrate briefly the method of the present section by a number of simple examples.

Example 1. *Exponential Resistance Gradient*, $\rho/\rho_s = e^{2mz}$, $(0 < m < \infty)$.

We easily find on solving (78), and taking that solution which vanishes as $z \to \infty$,

$$V = \frac{I\rho_s}{2\pi}\int_0^\infty \frac{e^{-(\mu-m)z}}{\mu - m}\lambda J_0(r\lambda)\,d\lambda, \qquad (82)$$

where $\mu = \sqrt{(\lambda^2 + m^2)}$.

To evaluate the integral, we make use of the well-known formula

$$I_1 = \int_0^\infty e^{-\mu z}\frac{\lambda}{\mu}J_0(r\lambda)\,d\lambda = \frac{e^{-mR}}{R}, \qquad (83)^*$$

where $R = \sqrt{(r^2 + z^2)}$.

If we multiply each side of (83) by r and integrate from 0 to r we find

$$I_2 = \int_0^\infty e^{-\mu z}\frac{J_1(r\lambda)}{\mu}\,d\lambda = \frac{1}{mr}\{e^{-mz} - e^{-mR}\}, \qquad (84)$$

and, differentiating with respect to z,

$$-\frac{\partial I_2}{\partial z} = \int_0^\infty e^{-\mu z}J_1(r\lambda)\,d\lambda = \frac{1}{r}\left\{e^{-mz} - \frac{z}{R}e^{-mR}\right\}. \qquad (85)$$

We now find from (82), after a few simple reductions,

$$-\frac{dV}{dr} = \frac{I\rho_s}{2\pi}e^{mz}\left\{m^2 I_2 - m\frac{\partial I_2}{\partial z} - \frac{\partial I_1}{\partial r}\right\}, \qquad (86)$$

from which the potential at any point (r, z) may be obtained by integration with respect to r.

On making $z \to 0$, we easily find

$$-\frac{dV_s}{dr} = \frac{I\rho_s}{2\pi}\left[\frac{m}{r}(1 - e^{-mr}) + \frac{m}{r} + \frac{e^{-mr}}{r^2} + m\frac{e^{-mr}}{r}\right], \qquad (87)$$

and hence, ultimately,

$$\bar{\rho}/\rho_s = 2mr + e^{-mr}, \qquad (88)$$

agreeing with the determination of Section 5.

Example 2. *Exponential Resistance Gradient*, $\rho/\rho_s = e^{-2mz}$, $(0 < m < \infty)$.

Taking the solution of (78) which vanishes as $z \to \infty$, we have

$$V = \frac{I\rho_s}{2\pi}\int_0^\infty \frac{e^{-(\mu+m)}}{\mu + m}\lambda J_0(r\lambda)\,d\lambda.$$

The procedure of Example 1 leads to the final result

$$\bar{\rho}/\rho_s = e^{-mr}. \qquad (89)$$

* WATSON, 13.47 (4).

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Example 3. *Resistance Gradients*, $\rho/\rho_s = (1 + z/a)^m$, $(0 < a < \infty)$, $(-\infty < m < \infty)$.

Equation (78) for Z is

$$\frac{d^2Z}{dz^2} - \frac{m}{a+z}\frac{dZ}{dz} - \lambda^2 Z = 0. \quad (90)$$

The solution of this which tends to zero as $z \to \infty$ is,

$$Z\{\lambda(a+z)\} = (a+z)^{\frac{1}{2}(m+1)} K_{\frac{1}{2}(m+1)}\{\lambda(a+z)\}.$$

With $\nu = \frac{1}{2}(m-1)$, as usual, we easily find that

$$V = \frac{I\rho_s}{2\pi}\left(1+\frac{z}{a}\right)^{\nu+1}\int_0^\infty \frac{K_{\nu+1}\{\lambda(a+z)\}}{K_\nu(\lambda a)} . J_0(\lambda r)\,d\lambda, \quad (91)$$

and

$$\psi = -\frac{I}{2\pi}\left(1+\frac{z}{a}\right)^{\nu}\int_0^\infty \frac{K_{\nu}\{\lambda(a+z)\}}{K_\nu(\lambda a)} . rJ_1(\lambda r)\,d\lambda, \quad (92)$$

while on writing $z = 0$ in (91), we have for the surface potential

$$V_s = \frac{I\rho_s}{2\pi}\int_0^\infty \frac{K_{\nu+1}(\lambda a)}{K_\nu(\lambda a)} . J_0(\lambda r)\,d\lambda. \quad (93)$$

On the axis of z, $r = 0$, and $\psi_0 = 0$, while for $z = 0$,

$$\psi_s = -I/(2\pi) . \int_0^\infty rJ_1(\lambda r)\,d\lambda = -I/(2\pi),$$

so that equation (3) is satisfied.

When m is an even integer (positive, negative, or zero), the ratio of the K-functions in the above formulæ becomes the ratio of polynomials in (λa), and sometimes the integral may be evaluated in terms of known functions. The following examples will suffice.

(i) $m = -2$, $\nu = -\frac{3}{2}$. On remembering that $K_{-\nu} = K_\nu$, equation (93) gives

$$V_s = \frac{I\rho_s}{2\pi}\int_0^\infty \frac{K_{\frac{1}{2}}(\lambda a)}{K_{\frac{3}{2}}(\lambda a)} J_0(r\lambda)\,d\lambda = \frac{I\rho_s}{2\pi}\int_0^\infty \frac{a\lambda}{1+a\lambda} J_0(\lambda r)\,d\lambda.$$

Thus, in consequence of (81), we have

$$\frac{\bar{\rho}}{\rho_s} = 1 - \int_0^\infty \frac{\lambda\, J_1(\lambda r)\,d\lambda}{1+a\lambda} = 1 + \frac{r^2}{a^2} + \frac{\pi}{2}\frac{r^2}{a^2}\{Y_1(r/a) - H_1(r/a)\}, \quad (94)$$

in agreement with (24), and in the derivation of which we have made use of the formula

$$\int_0^\infty \frac{x^\nu J_\nu(ax)}{x+k}\,dx = \frac{\pi k^\nu}{2\cos\pi\nu}\{H_{-\nu}(ak) - Y_{-\nu}(ak)\}, \quad (95)^*$$

vaild for $-\frac{1}{2} < R(\nu) < \frac{3}{2}$, and $0 < R(k) < \infty$.

* WATSON, § 13.6 (7).

(ii) $m = 0$, $\nu = -\frac{1}{2}$. We have

$$V_s = I\rho_s/(2\pi) \,.\, \int_0^\infty J_0(\lambda r)\, d\lambda = I\rho_s/(2\pi)\,.\, r^{-1},$$

so that $\bar{\rho}/\rho_s = 1$.

(iii) $m = 2$, $\nu = \frac{1}{2}$. Here

$$V_s = \frac{I\rho_s}{2\pi}\int_0^\infty \frac{K_{\frac{3}{2}}(\lambda a)}{K_{\frac{1}{2}}(\lambda a)}\,.\, J_0(\lambda r)\, d\lambda = \frac{I\rho_s}{2\pi}\int_0^\infty \left(1 + \frac{1}{\lambda a}\right) J_0(\lambda r)\, d\lambda,$$

giving $\bar{\rho}/\rho_s = 1 + r/a$, in agreement with formulæ (47) and (71).

In general, the identity of the expressions for ψ given in (92) with those obtained in Section 6 can be established as a particular case of a theorem in contour integration.*

For purposes of graphical computation, the integrals in formulæ (61) and (66) for $\bar{\rho}/\rho_s$ are more convergent than those in the corresponding formulæ derived from (93),

$$\frac{\bar{\rho}}{\rho_s} = r^2 \int_0^\infty \frac{K_{\nu+1}(\lambda a)}{K_\nu(\lambda a)}\,.\, \lambda\, J_1(\lambda r)\, d\lambda. \quad \ldots \ldots \ldots \quad (96)$$

By applying HANKEL's inversion theorem to (101), we have

$$\frac{K_{\nu+1}(\lambda a)}{K_\nu(\lambda a)} = \int_0^\infty \frac{\bar{\rho}}{\rho_s}\, \frac{J_1(\lambda r)}{r}\, dr.$$

If we now introduce for $\bar{\rho}/\rho_s$ the solutions (61) and (66), interchange the order of integration and make use of the integral

$$\int_0^\infty t K_1(at)\,.\, J_1(bt)\, dt = (b/a)\,.\,(a^2 + b^2)^{-1}. \quad \ldots \ldots \ldots \quad (97)\dagger$$

we are led to the interesting result

$$\frac{K_{\nu+1}(\lambda a)}{K_\nu(\lambda a)} = \frac{2\nu}{\lambda a} + \frac{4\lambda}{\pi^2 a}\int_0^\infty \frac{dt}{\{Y_\nu^2(at) + J_\nu^2(at)\}(t^2 + \lambda^2)\, t}, \quad (0 < \nu < \infty). \quad . \quad . \quad (98)$$

In the interval $-\infty < \nu < 0$, the first term must be omitted.

This result is typical of a large number of interesting integrals which may be established by comparing the ψ- and V-solutions.

Example 4. *Resistance Gradient*, $\rho/\rho_s = (1 + z/a)^{-2}$.

It is easily shown that equation (77) for $Z(\lambda, z)$ may be solved in the form $Z = v\sqrt{\rho/\rho_s}$, provided that v satisfies the equation,

$$\frac{d^2 v}{dz^2} - \left(\lambda^2 + \tfrac{1}{2}\frac{dP}{dz} + \tfrac{1}{4}P^2\right)v = 0, \quad \text{where} \quad P = -\frac{1}{\rho}\frac{d\rho}{dz}. \quad \ldots \ldots \quad (99)$$

* Appendix I of this paper, Example 2.

† WATSON, § 13.45 (2).

For the resistance gradient of the present example, we thus find $Z = e^{-\lambda z}/(a + z)$, so that

$$V = \frac{I\rho_s}{2\pi}\int_0^\infty \frac{a^2\lambda}{1 + a\lambda} \cdot \frac{e^{-\lambda z}}{(a + z)} J_0(\lambda r)\, d\lambda, \qquad (100)$$

and in particular

$$\frac{\bar{\rho}}{\rho_s} = 1 - \int_0^\infty \frac{\lambda\, J_1(\lambda r)\, d\lambda}{1 + a\lambda}, \qquad (101)$$

agreeing with (94).

Example 5. *Resistance Gradient*, $\rho/\rho_s = \cosh^2 ma \,.\, \text{sech}^2\, m(z - a)$, $(0 < a < \infty)$.

This example is of some practical interest as representing a case where ρ/ρ_s increases to a maximum at $z = a$, and falls off to zero as $z \to \infty$. The method of the above example gives $Z = e^{-z\sqrt{(\lambda^2+m^2)}}\, \text{sech}\, m(z - a)$, and

$$V = \frac{I\rho_s}{2\pi} \cosh ma \,.\, \text{sech}\, m(z - a) \,.\, \int_0^\infty \frac{e^{-z\sqrt{(\lambda^2+m^2)}}}{\sqrt{(\lambda^2 + m^2)} - m \tanh ma} \,.\, \lambda\, J_0(r\lambda)\, d\lambda. \qquad (102)$$

By writing $z = 0$ in the above expression, we at once obtain the surface potential V_s, but the result is not easily integrated. It may, however, by the use of Sonine's formula,* be expressed as a convergent series of K_n-functions by expanding the integrand in powers of $m \tanh ma/(\lambda^2 + m^2)^{\frac{1}{2}}$, and integrating term by term.

Example 6. *Resistance Gradient*, $\rho/\rho_s = \text{sech}^2\, mz$.

By writing $a = 0$ in (102) we obtain a simple result

$$V_s = \frac{I\rho_s}{2\pi}\int_0^\infty \frac{\lambda J_0(\lambda r)}{(\lambda^2 + m^2)^{\frac{1}{2}}}\, d\lambda = \frac{I\rho_s}{2\pi}\frac{e^{-|m|r}}{r} \qquad (103)\dagger$$

and

$$\bar{\rho}/\rho_s = (1 + |m|/r)\, e^{-|m|r}. \qquad (104)$$

Section 9. *Geophysical Applications. Determination of gradient of resistance from surface-gradient characteristic.*

It will be noticed from the particular examples worked out in detail in the preceding sections, that if we expand the resistance gradient in the form

$$\rho = \rho_s(1 + 2mz + \ldots), \qquad (105)$$

the corresponding expansion of the "surface-gradient characteristic" in powers of r, the distance from the electrode, is always of the form

$$\bar{\rho} = \rho_s(1 + mr + \ldots). \qquad (106)$$

* Watson, § 13.6 (2).

† Watson, § 13.6 (2), with $\nu = 0$ and $\mu = -\frac{1}{2}$.

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This rule is found to hold when the medium is bounded by a perfectly conducting or insulating plane, as well as in examples in which ρ is finite and continuous throughout the interval $0 < z < \infty$.

The proof of this result may be inferred from Examples 1 and 2 of Section 8, since the resistance gradient (105) may be represented approximately by the exponential gradient $\rho/\rho_s \sim e^{2mz}$. The expansion of (88) or (89) corresponding to negative or positive values of m leads at once to (106) as the corresponding approximations to $\bar{\rho}/\rho_s$.

[*Added February* 12, 1934.—Since the author's paper was first communicated, there has appeared an interesting method due to R. E. LANGER* for deriving the expansion of ρ/ρ_s in powers of z when $\bar{\rho}/\rho_s$ is given as a function of r. Assuming that there is no discontinuity in the resistance-gradient, the expansion is shown to be unique. Briefly stated, the procedure referred to is as follows :—

The theory of Section 8 shows that the surface-potential may be expressed by the integral

$$V_s = \frac{I\rho_s}{2\pi}\int_0^\infty F(\lambda)\, J_0(\lambda r)\, d\lambda, \quad \ldots \ldots \ldots \ldots \quad (107)$$

from which it follows by (4) that

$$\frac{1}{r^2}\frac{\bar{\rho}}{\rho_s} = \int_0^\infty F(\lambda)\, \lambda J_1(\lambda r)\, d\lambda,$$

and hence by HANKEL's inversion theorem,

$$F(\lambda) = \int_0^\infty \frac{\bar{\rho}}{\rho_s} J_1(\lambda r)\, \frac{dr}{r}. \quad \ldots \ldots \ldots \ldots \quad (108)$$

Should it be possible to expand F (λ) in the form

$$F(\lambda) = 1 + \frac{a_1}{\lambda} + \frac{a_2}{\lambda^2} + \ldots + \frac{a_n}{\lambda^n} + \ldots, \quad \ldots \ldots \ldots \quad (109)$$

for large values of λ, LANGER's theorem states that the law of specific resistance with depth is given by the expansion

$$\tfrac{1}{2}\log\frac{\rho(z)}{\rho_s} = \alpha_1 z + \frac{\alpha_2}{2\,!}z^2 + \frac{\alpha_3}{3\,!}z^3 + \frac{\alpha_4}{4\,!}z^4 + \ldots \ldots \ldots \quad (110)$$

In terms of a's of the expansion (109),

$$\left.\begin{aligned} \alpha_1 = a_1. \quad \alpha_2 = 2a_2 - a_1^2 \quad \alpha_3 = 4a_3 - 4a_1a_2 + 2a_1^3 \\ \alpha_4 = 2\{8a_1^2a_2 - 3a_1^4 - 4a_1a_3 - 2a_2^2 + 4a_4\} \end{aligned}\right\} \quad \ldots \quad (111)$$

* 'Bull. Amer. Math. Soc.,' p. 814 (October, 1933). See also SLICHTER, 'Physics,' vol. 4, p. 307 (1933).

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To determine the coefficient α_n, we compute

$$\phi_1 = 2a_2 - a_1^2, \quad \phi_2 = 2a_3. \quad \phi_n = 2a_{n+1} + \sum_{s=1}^{n-2} a_{s+1}a_{n-s}. \quad . \quad . \quad . \quad . \qquad (112)$$

It is then proved by LANGER that

$$\alpha_n = \sum_{r=1}^{n} \phi_r \frac{\partial \alpha_{n-1}}{\partial a_r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (113)$$

We easily find from (110) and (111) that in terms of the coefficients of (109),

$$\frac{\rho(z)}{\rho_s} = 1 + 2a_1z + 2(a_2 + \tfrac{1}{2}a_1^2)z^2 + \tfrac{4}{3}(a_3 + 2a_1a_2)z^3$$
$$+ \tfrac{2}{3}(a_4 + 3a_1a_3 + a_1^2a_2 + \tfrac{5}{2}a_2^2)z^4 + \ldots \ldots \qquad (114)$$

In order to determine the a's of the above expansion from a field-graph of $\bar{\rho}/\rho_s$, the writer of the present paper suggests that the curve be fitted, if possible, to an expression of the form

$$\bar{\rho}/\rho_s = 1 + mr + \sum_s A_s e^{-m_s r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (115)$$

The terms under the summation represent the ordinates intercepted by the curve and a parallel to the asymptote through $\bar{\rho}/\rho_s = 1$. The constants A_s and m_s may then be determined by PRONY's method for expressing a function tabulated at equal intervals by a sum of exponentials. From (108) we obtain

$$F(\lambda) = 1 + \frac{m}{\lambda} + \sum_s A_s \left\{\frac{m_s}{\lambda} + \left(1 + \frac{m_s^2}{\lambda^2}\right)^{\frac{1}{2}}\right\},$$

and since $\sum_s A_s = 0$, we have

$$a_1 = m + \Sigma A_s m_s, \quad a_2 = \tfrac{1}{2}\Sigma A_s m_s^2, \quad a_3 = 0, \quad a_4 = -\tfrac{1}{8}\Sigma A_s m_s^4. \quad . \qquad (116)$$

The graph of $\rho(z)/\rho_s$ plotted against z according to (114) may be expected to reveal features of interest without undue labour of computation.* The solution here dealt with assumes $\rho(z)$ to be a continuous function of the depth throughout the entire interval $0 < z < \infty$, and does not contemplate discontinuities. A method of analysis suitable for discontinuous stratifications is briefly described in Section 11.]

First Order Correction to Depth Determination of Conducting Stratum Covering Insulating Bed Rock due to Surface Resistance Gradient.

It is evident from the general solution (19) that the graph of $\bar{\rho}/\rho_s$ plotted against r has an asymptote whose equation is

$$\frac{\bar{\rho}}{\rho_s} = r \Big/ \int_0^h \frac{\rho_s}{\rho}\, dz. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (117)$$

* [*Added May* 12, 1934—This point has recently been examined for known discontinuous stratifications by A. F. STEVENSON, 'Physics,' vol. 5, p. 14, April, 1934.]

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It is important to note that the existence of an asymptote as shown in fig. 1 is in itself no evidence of the existence of an electrically insulating stratum, since such an asymptote will exist in a semi-infinite medium for which $\int_0^\infty (\rho_s/\rho)\, dz$ remains finite.

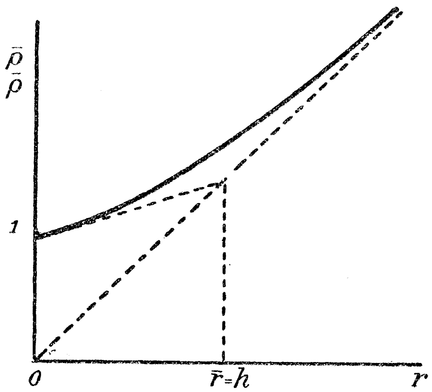

FIG. 1.

If, however, it is certain from geological evidence that such a stratum does exist at a depth which it is not impossible to determine by measurements of surface-potential gradients, the results of the present section lead to a simple practical method of correcting for a possible resistance gradient near the surface.

Substituting the approximate expansion of ρ/ρ_s from (105) in (117), and retaining only first powers of mh, we have for the asymptote the approximate equation

$$\bar{\rho}/\rho_s \sim (r/h)(1 + mh + \ldots),$$

while, according to (106), the equation of the tangent at the point $(\bar{\rho}/\rho_s) = 1$ is $\bar{\rho}/\rho_s \sim 1 + mr$. Solving for the r-co-ordinate of the intersection, $\bar{r}$, we find at once, $h = \bar{r}$.

As a matter of practice in field measurements for depth determination of an insulating stratum, it is important to take a sufficient number of surface-potential gradients near the electrode to determine the graph $\bar{\rho}/\rho_s$ near $r = 0$ with such accuracy that by methods of equal ordinates the slope of the tangent at the origin may be determined.* The position of the asymptote may be determined by measuring the surface-potential gradients at a distance of several times the suspected depth of the bed rock, preferably for a number of azimuths; the simple rule just enunciated will then give the depth corrected for a possible gradient of specific resistance.

Derivation of Surface Gradient Characteristic $\bar{\rho}/\rho_s$ from Two-Electrode Observations.

The analysis of the present paper is confined to the problem of determining the electrical depth constants from surface-potential gradients due to a single electrode. In practice it is, of course, necessary to employ at least two electrodes whose distance apart should be several times as great as the depth which it is desired to explore. In general, the direct interpretation of two-electrode gradients is not easy. We proceed,

* Such a formula as that due to GREGORY and NEWTON,

$$f'(a) = (1/w)\{\Delta f(a) - \tfrac{1}{2}\Delta^2 f(a) + \tfrac{1}{3}\Delta^3 f(a) - \tfrac{1}{4}\Delta^4 f(a) + \ldots\}$$

may be employed in which w is the equal interval and Δ, Δ^2, Δ^3, . . . successive differences of a function $f(a)$. (WHITTAKER and ROBINSON, "The Calculus of Observations," p. 35 (Blackie and Son (1924).)

therefore, to show how to obtain the surface-gradient characteristic $\bar{\rho}/\rho_s$ from two-electrode observations, since the corresponding graph is much more easily analysed and interpreted in terms of the depth constants. We suppose that observations are confined to points in the line of the electrodes distant $2l$ apart, as a result of which it is possible to plot a graph of dV_s/dr against distance from one of the electrodes, fig. 2.

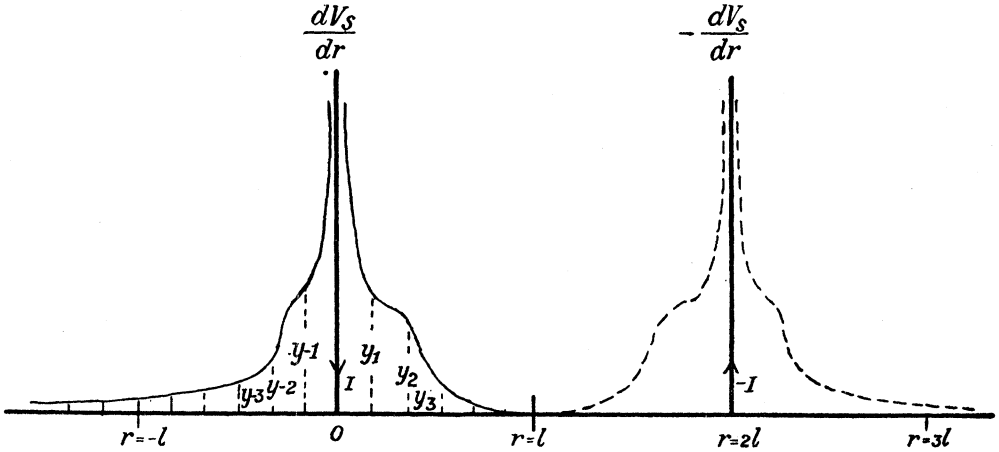

FIG. 2.

If the specific resistance is everywhere a function of the depth only, the graph will be symmetrical about an axis midway between the electrodes if r is measured in a positive sense from each electrode; and this symmetry is at once a test of the applicability of the present theory. We therefore confine our attention to the left-hand portion of the curve which may be plotted as an average of the two.

Measure r from the electrode O. Then, since I takes the opposite sign at the second electrode, the surface potential due to the two electrodes at a point r along the line joining them is

$$V_s = V_1(r) - V_1(2l - r),$$

where $V_1(r)$ is the surface potential due to one electrode.

Since

$$\frac{dV_s}{dr} = \frac{d}{dr} V_1(r) - \frac{d}{dr} V_1(2l - r), \quad \dots\dots\dots \quad (118)$$

the problem before us is to determine dV_1/dr from a series of known values of dV_s/dr. Evidently dV_1/dr is a function of r only, which we may conveniently denote by $dV_1/dr = f(r)$, so that (118) gives

$$\frac{dV_s}{dr} = f(r) - f(2l - r). \quad \dots\dots\dots \quad (119)$$

Since we consider r to be measured in the positive sense, $f(r) = f(-r)$, where $-r$ refers to a point on that side of the electrode under consideration remote from the second.

We now divide up the interval 0 to l into equal intervals h, so that values of $d\mathrm{V}_s/dr$ may be read off from the field graph as the ordinates $y_1, y_2, \ldots y_{n-1}$.

On making use of the convenient notation $f(sh) = f_s$, equation (119) applied to the values of $d\mathrm{V}_s/dr$ at $r = h, 2h, \ldots (n-1)h$, gives at once $n - 1$ equations of the type

$$f_s - f_{2n-s} = y_s, \quad (s = 1, 2, \ldots n-1). \quad \ldots \ldots \ldots \quad (120)$$

Regarding these as a set of simultaneous equations for the determination of $f_1, f_2, \ldots f_{2n-s}$, it is obvious that we have only $n - 1$ equations for $2n - 2$ variables. We therefore make use of observations on the side of the electrode remote from the second.

Remembering that $f(r) = f(-r)$, or $f(sh) = f(-sh)$, or $f_s = f_{-s}$ we have a second set of n equations of the type

$$f_s - f_{2n+s} = y_{-s}, \ (s = 1, 2, \ldots n), \quad \ldots \ldots \ldots \quad (121)$$

in which there enter only $n + 1$ new variables. The set of equations (120) and (121) is still insufficient in number. We therefore separate the electrodes to a distance $4l$ and obtain a second graph from which the ordinates $z_1, z_2, \ldots z_{2n-1}$ at equal intervals h may be read off. We now obtain another set of $2n - 1$ equations of the type

$$f_s - f_{4n-s} = z_s, \quad (s = 1, 2, \ldots 2n-1), \quad \ldots \ldots \quad (122)$$

in which there has now been introduced $n - 1$ new variables. The three sets (120), (121) and (122) now give $4n - 2$ equations for $4n - 2$ variables which may be solved to give numerical values of $f_s = f(sh)$ at equal intervals h for values of s from 1 to $4n - 1$, with the omission of f_{2n}, which does not enter into the equations. As some of the equations are redundant, it is necessary to use observations for electrode separation $4l$ along the line joining them remote from either.

Since $\bar{\rho}/\rho_s = -(2\pi/\mathrm{I})r^2\, d\mathrm{V}_s/dr = -(2\pi/\mathrm{I})(sh)^2 f_s$, we are able to construct the graph for the surface-gradient characteristic due to a single electrode in a form suitable for comparison with theoretical values, or for analysis by graphical integration, making use of HANKEL's inversion theorem in the manner described in the writer's first paper.* As the influence of the lower layers is shown up in the graph of $\bar{\rho}/\rho_s$ for values of r of the order of the depth, the necessity of working with as large electrode separations as possible is evident.

Section 10. *Determination of Surface Potentials for Line-Electrodes.*

If, for practical reasons, it is found desirable to minimize the hazard of local irregularities of specific resistance near the electrode by employing a continuous line electrode, the results of the present paper can be used without the need of constructing a special

* 'Proc. Roy. Soc.,' A, vol. 139, p. 262 (1933).

theory. The theoretical procedure is to integrate the expression for the surface potential due to current w (s) entering the medium for an element ds of the line electrode. If not given explicitly, the surface potential is easily obtained by integrating equation (4),

$$-dV_s/dr = I/(2\pi) \,.\, (\bar{\rho}/r^2).$$

In general, it follows from equation (19), that for media bounded by perfectly conducting or insulating planes the expression for V_s is of the form

$$-V_s = I/(2\pi) \,.\, \{B \log r - \sum_s B_s K_0 (r\lambda_s)\}, \quad \ldots \ldots \quad (123)$$

in which the depth-constants are contained in the B's and λ's, the latter being the roots of an equation of condition. When the specific resistance is continuous throughout an infinite depth from the surface, or from the last plane of discontinuity of ρ, the summation must be replaced by a definite integral.

We proceed to consider a few simple examples in which V_s may be integrated along a line electrode.

Example 1. *Circular Line Electrode of Radius a.*

If w is the current introduced per unit length of electrode, V_s the required surface potential at distance r from the centre of the circle, an element at (a, θ) introduces current $wad\,\theta$, so that an expression of the type (123) requires us to evaluate the integrals in the formula

$$-V_s = (\tfrac{1}{2}wa/\pi)\Big[B\int_0^{2\pi} \log (r^2 + a^2 - 2ar \cos \theta)^{\frac{1}{2}} \, d\theta$$
$$-\sum_s B_s \int_0^{2\pi} K_0\{\lambda_s (r^2 + a^2 - 2ar \cos \theta)^{\frac{1}{2}}\} d\theta\Big]. \quad \ldots \quad (124)$$

The first integral is $2\pi \log a$ or $2\pi \log r$ according as $r <$ or $> a$. In view of the expansion, ($|\,r\,| < |\,a\,|$),

$$K_0 \{\lambda_s (r^2 + a^2 - 2ar \cos \theta)^{\frac{1}{2}}\} = I_0 (\lambda_s r) \,.\, K_0 (\lambda_s a) + 2\sum_{m=1}^{\infty} I_m (\lambda_s r) K_m (\lambda_s a) \cos m\theta \quad (125)^*$$

the second integral is

$$2\pi \, I_0 (\lambda_s r) \,.\, K_0 (\lambda_s a), \text{ or } 2\pi \, I_0 (\lambda_s a) \,.\, K_0 (\lambda_s r)$$

according as $r <$ or $> a$.

Thus

or

$$\left.\begin{aligned} -dV_s/dr &= wa \sum_s B_s \lambda_s I_1 (\lambda_s r) \,.\, K_0 (\lambda_s a), \quad (0 < r < a), \\ -dV_s/dr &= wa/r + wa \sum_s B_s \lambda_s K_1 (\lambda_s r) \,.\, I_0 (\lambda_s a), \quad (a < r < \infty) \end{aligned}\right\} . \quad . \quad (126)$$

* WATSON, § 11.41 (8).

We may employ this procedure in the various examples worked out in the present paper where the summation is replaced by a definite integral, and $J_0(\lambda r)$ or $K_0(\lambda r)$ by the products of two Bessel functions. The results are not, however, easily integrable and are of no special interest.

We may combine an expression derived from (123) with (126) by writing $I = -2\pi wa$ to obtain the surface potential due to a central electrode and a concentric circular electrode at a great distance. It is evident, however, that the derivation of the B's and λ's from the resulting expressions is a problem presenting formidable difficulties.

Example 2. *Infinite Straight Line Electrode.*

If V_s is the required surface potential at a perpendicular distance x from the electrode, while the current entering the medium at a distance ζ along the electrode from the foot of the perpendicular is $wd\zeta$, equation (123) leads to the expression

$$-V_s = \frac{w}{2\pi}\left[B\int_{-\infty}^{\infty}\log(x^2+\zeta^2)^{\frac{1}{2}}\,d\zeta - \sum_s B_s\int_{-\infty}^{\infty}K_0\{\lambda_s(x^2+\zeta^2)^{\frac{1}{2}}\}\,d\zeta\right]. \quad (127)$$

It is more convenient to deal with $-dV_s/dx$ which gives

$$-\frac{dV_s}{dx} = \frac{w}{\pi}\left[\tfrac{1}{2}\pi B + \sum_s B_s\lambda_s x\int_0^{\infty}\frac{K_1\{\lambda_s(x^2+\zeta^2)^{\frac{1}{2}}\}\,d\zeta}{(x^2+\zeta^2)^{\frac{1}{2}}}\right] \quad . \quad . \quad . \quad . \quad (128)$$

In calculating surface-potential gradients for the infinite straight line electrode, we note the two integrals

$$\int_0^{\infty}\frac{K_1\{\lambda(x^2+\zeta^2)^{\frac{1}{2}}\}}{(x^2+\zeta^2)^{\frac{1}{2}}}\,d\zeta = \frac{\pi}{2}\frac{e^{-\lambda x}}{\lambda x} \quad \text{and} \quad \int_0^{\infty}J_0\{\lambda(x^2+\zeta^2)^{\frac{1}{2}}\}\,d\zeta = \frac{\cos\lambda x}{\lambda}, \quad (129)^*$$

so that (128) takes the form

$$-\frac{dV_s}{dx} = \tfrac{1}{2}w\{B + \sum_s B_s e^{-\lambda_s x}\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130)$$

Occasionally the various examples of resistance gradients lead to comparatively simple results which the reader will have no difficulty in proving.

(i) *Medium of Constant Specific Resistance*, $\rho/\rho_s = 1$. We immediately obtain

$$-dV_s/dx = w\rho_s/(\pi x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

(ii) *Exponential Resistance Gradient*, $\rho/\rho_s = e^{2mz}$.—The methods of Sections 5 and 8 agree in giving

$$-\frac{dV_s}{dx} = w\rho_s\left\{m + \frac{m}{\pi}\int_x^{\infty}\frac{K_1(mx)}{x}\,dx\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)$$

* WATSON, § 13.47 (6) and § 13.47 (5).

(iii) *Exponential Resistance Gradient*, $\rho/\rho_s = e^{-2mz}$.—In this case

$$-\frac{dV_s}{dx} = w\rho_s \frac{m}{\pi}\int_x^\infty \frac{K_1(mx)}{x}\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (133$$

(iv) *Parabolic Resistance Gradient*, $\rho/\rho_s = (1 + z/a)^2$.—It is easily proved that

$$-\frac{dV_s}{dx} = \frac{w\rho_s}{\pi}\left(\frac{1}{x} + \frac{\pi}{2a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (135)$$

We may note that all the results for an infinite straight line electrode may be independently derived from a two-dimensional version of the theory of the present paper; it is, however, unnecessary to develop this theory, when surface potentials for point electrodes are known.

It will be noticed that from the theoretical point of view the expressions for surface potential gradients due to line electrodes are never simpler than in the theory already discussed for point electrodes. This is to be expected since the depth constants are contained in differential equations of the type (7) or (78) which remain the same in the two- and symmetrical three-dimensional theory.

Section 11. *Note on the Determination of the Depth Constants.*

When a straight line electrode is of finite length, as it must be in practice, the integrals in (127) must be taken between finite limits $\pm l$, and cannot easily be evaluated. It is to be noted, however, that (130), expressed as a series of exponentials, is considerably simpler as a subject for analysis than the corresponding expression for the point electrode, involving the functions $K_1(\lambda_s r)$. We may write (1·30) in the form

$$\bar{\rho}/r^3 = F(r) = B/r^2 + \sum_s B_s\lambda_s K_1(\lambda_s r)\,.\,/r,$$

where the left-hand side may be supposed known from field observations. If we now replace r^2 by $x^2 + \zeta^2$, and integrate with respect to ζ between 0 and ∞, we have by (129)

$$x\int_0^\infty F\{(x^2+\zeta^2)^{\frac{1}{2}}\}\,d\zeta = \tfrac{1}{2}\pi\left[B + \sum_s B_s e^{-\lambda_s x}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (136)$$

in which the left-hand side may be computed by graphical integration for a series of values of x, preferably at equal intervals, from the known graph of $F(r)$ plotted against r. This means that from a series of observations with two electrodes, it is possible to compute the surface potential gradients characteristic of an infinite straight line electrode. The advantage in doing so is that it now becomes theoretically possible to determine the B's and λ's by PRONY's method* of expressing a function tabulated at equal intervals

* WHITTAKER and ROBINSON, "Calculus of Observations," § 180, p. 369.

by a sum of exponentials. Should the method prove to be practical in spite of somewhat heavy arithmetical labour, the results, interpreted by theory would lead to the possibility of determining the electrical depth constants from the results of field observations. If, for instance, the λ's should tend to equality, the inference is that the resistance gradient tends to be a continuous function of the depth, theoretically determinable from a possible solution of LAPLACE's integral equation, $f(x) = \int_0^\infty e^{-xt}\phi(t)\,dt$. Should the λ's be widely separated in value, it may be concluded that the resistance gradient shows discontinuity, and there is a possibility of computing the stratification constants, thickness, specific resistance, and depths of the discontinuous layers from the numerical values of the B's and λ's computed by PRONY's method of "exponential analysis" just cited. The further development of this procedure is best dealt with in connection with the general theory of multiply-stratified media which the writer hopes to deal with in a future communication.

Section 12. *Summary and Conclusions.*

(1) The theory of current-flow in a semi-infinite medium in which the specific resistance is a continuous function of the depth is developed for the special case where the current is introduced at a single surface electrode.

(2) Using the electrical stream-function ψ, the solution depends on a differential equation of the Sturm-Liouville type.

(3) The theory of Sturm-Liouville expansions, especially adapted to the single-electrode problem, is developed for a medium bounded by a perfectly insulating or conducting plane parallel to the surface.

(4) In the limiting case when the boundary plane is at an infinite distance, the expansion for ψ becomes a definite integral into which enter determinate factors depending on the assigned law of variation of specific resistance with depth.

(5) The calculation of the surface potential is carried out for a number of simple laws of variation of specific resistance with depth, often leading to integrable results.

(6) The resistance gradients $\rho/\rho_s = (1 + z/a)^m$, in which m is unrestricted, are worked out in some detail as an application of the general theory.

(7) The general theory for the direct determination of the potential V is outlined. Although the resulting values are of different form to those obtained by the use of the stream-function ψ, they are shown to be related by a general theorem involving contour integration.

(8) Several resistance gradients are worked out in terms of the potential V and the results are found to agree with previous calculations.

(9) Applications to practical geophysical prospecting are considered in some detail. In particular, it is shown how the depth determination of an insulating stratum may be corrected for a surface gradient of specific resistance.

(10) A method is suggested for taking field observations in such a way that from two-electrode surface potential gradients, the "surface-gradient characterictic" due to a single electrode may be obtained. The latter is better suited to analysis for the determination of the depth constants.

(11) A few examples are worked out for surface potentials from continuous line electrodes, such as the infinite line and the circle. In general, it is found that the results are more difficult to interpret than those from the single electrode.

(12) By a series of graphical integrations, making use of the single-electrode "surface-gradient characteristic," it is shown how the potential gradients due to an infinite straight line electrode may be obtained. The latter are especially well suited to a process of exponential analysis, and to the ultimate determination of the electrical depth constants.

Appendix I.

Note on Integrals of the Form $\int_0^\infty \frac{\phi(z)}{K_\nu(z)}\,dz, \quad (-\infty < \nu < \infty)$.

It is known that the function $K_\nu(z)$ has no zeros on the positive portion of the real axis, no purely imaginary zeros, and no zero within the quadrant corresponding to $0 < \arg z < \frac{1}{2}\pi$.* A large number of interesting integrals may be derived by integrating $\phi(z)/K_\nu(z)$ along a contour formed by the positive parts of the real and imaginary axes with a quadrantal arc (in the first quadrant) of a circle whose radius is made to tend to infinity, the contour having a quadrantal indentation at the origin.

Example 1. In particular, consider

$$\int \frac{K_\nu(zt)}{K_\nu(z)}\frac{dz}{z}, \quad \text{(i)}$$

where $(t-1)$ is positive and real, in which case the asymptotic formula for $K_\nu(zt)$ shows that the contribution of the infinite quadrant is zero. Near the origin

$$K_\nu(z) \sim \tfrac{1}{2}\,.\,(\tfrac{1}{2}z)^{-|\nu|}\,.\,\Pi(|\nu|-1),$$

and when $\nu = 0$,

$$K_0(z) \sim (\gamma + \log \tfrac{1}{2}z), \quad \text{(ii)}$$

where the positive value of ν, denoted by $|\nu|$, is taken since

$$K_{-\nu}(z) = K_\nu(z). \quad \text{(iii)}$$

It follows at once that the contribution due to the indentation is

$$\tfrac{1}{2}\pi i t^{-|\nu|}. \quad \text{(iv)}$$

Along the imaginary axis, $z = iy$, and

$$K_\nu(iy) = \tfrac{1}{2}\pi i\, e^{\frac{1}{2}\pi i\nu}\, H_0^{(1)}(-y) = \tfrac{1}{2}\pi i\, e^{-\frac{1}{2}\pi i\nu}\{J_\nu(y) - i\, Y_\nu(y)\}. \quad \text{(v)†}$$

* Watson, § 15.7.

† Watson, § 3.7 (8) and § 3.62 (5).

Since there are no zeros in the contour, we may write

$$\int_0^\infty \frac{K_\nu(xt)}{K_\nu(x)}\frac{dx}{x} - \int_0^\infty \frac{J_\nu(yt) - iY_\nu(yt)}{J_\nu(y) - iY_\nu(y)}\frac{dy}{y} = \tfrac{1}{2}\pi i t^{-|\nu|}.$$

Equating real and imaginary parts, we have, in particular

$$\int_0^\infty \frac{Y_\nu(yt)\,J_\nu(y) - J_\nu(yt)\,Y_\nu(y)}{Y_\nu^2(y) + J_\nu^2(y)}\frac{dy}{y} = \tfrac{1}{2}\pi t^{-|\nu|}, \quad (1 < t < \infty). \quad \text{(vi)}$$

a result which is easily seen to hold for $\nu = 0$.

The integral is obviously discontinuous at $t = 1$, and for $\nu = \pm\frac{1}{2}$ is easily seen to be equivalent to

$$\int_0^\infty \sin\{(t-1)\,y\}.\frac{dy}{y} = \tfrac{1}{2}\pi, \quad (1 < t < \infty).$$

With a slight change of notation, this is the integral (58) used in Section 6.

Example 2. Consider $\int_0^\infty \frac{K_{\nu+1}(zt)}{K_\nu(z)}\,.\,H_0^{(1)}(rz)\,dz$ around the same contour as in Example 1 above. As before when t is real and greater than 1, the infinite quadrant contributes nothing to the integral.

Since $H_0^{(1)}(rz) = J_0(rz) + i\,Y_0(rz)$, near the origin

$$H_0^{(1)}(rz) \sim 1 + i\,(2/\pi)\,\{\gamma + \log\tfrac{1}{2}r + \log z), \quad \text{(vii)}$$

from which it follows, making use of (ii), that near the origin

$$\frac{K_{\nu+1}(zt)}{K_\nu(z)}\,H_0^{(1)}(rz) \sim \begin{cases} (2\nu/z)\,t^{-(\nu+1)}\{1 + i\,(2/\pi)\,(\gamma + \log\tfrac{1}{2}r + \log z)\}, & (0 < \nu < \infty) \\ (2i/\pi)/(zt), & (\nu = 0) \\ \tfrac{1}{2}\nu z\,t^{-(\nu+1)}\{1 + i\,(2/\pi)\,(\gamma + \log\tfrac{1}{2}r + \log z)\}, & (-\infty < \nu < 0). \end{cases} \quad \text{(viii)*}$$

Thus, since there are no zeros of $K_\nu(z)$ in the contour, we have, making use of (v) and $H_0^{(1)}(iry) = -(2/\pi)\,iK_0(ry)$, the result

$$\int_0^\infty \frac{K_{\nu+1}(xt)}{K_\nu(x)}\{J_0(rx) + iY_0(rx)\}\,dx + i\int_0^\infty \frac{J_{\nu+1}(yt) - iY_{\nu+1}(yt)}{J_\nu(y) - iY_\nu(y)}\,\frac{2}{\pi}\,K_0(ry)\,dy = \begin{cases} \pi i\nu t^{-(\nu+1)}\{1 + (2i/\pi)\,(\gamma + \log\tfrac{1}{2}r)\} \\ -1/t \\ 0. \end{cases}$$

* WATSON, § 3.6 (1), § 3.51 (2), § 3.54 (2).

Equating real and imaginaries, we have, in particular

$$-\int_0^\infty \frac{K_{\nu+1}(xt)}{K_\nu(x)} J_0(rx)\,dx = \frac{2}{\pi}\int_0^\infty \frac{\{Y_{\nu+1}(yt)\,J_0(y) - J_{\nu+1}(yt)\,Y_0(y)\}}{Y_\nu^2(y) + J_\nu^2(y)} K_0(ry)\,dy + \begin{cases} 2\nu t^{-(\nu+1)}(\gamma + \log \tfrac{1}{2}r) \\ 1/t \\ 0 \end{cases}, \quad . \; . \; . \quad \text{(ix)}$$

according as ν is positive, zero, or negative respectively.

With a slight change of notation the result (ix) identifies the expressions (59) and (96) for the potentials obtained by methods of Sections 6 and 8. It will be noticed that as derived in these sections the potentials may differ by a constant of integration whose value is thus deduced by contour integration.

Example 3. When $t = 1$, the convergence of the integral of Example 2 on the infinite quadrant fails. We note, however, that $\int \frac{\{K_{\nu+1}(z) - K_\nu(z)\}}{K_\nu(z)} . H_0^{(1)}(rz)\,dz$ is convergent on the infinite quadrant, which contributes nothing since the integrand is of the order $z^{-(3/2)}$ for large values of z. Along the real and imaginary axes we may take the integral in two portions. If we make use of the known integrals

$$\int_0^\infty J_0(rx)\,dz = 1/r, \qquad \int_0^\infty Y_0(rx) = 0, \qquad \int_0^\infty K_0(ry)\,dy = \tfrac{1}{2}\pi/r, \quad . \; . \; . \quad \text{(x)*}$$

we easily find that

$$\int^\infty \frac{K_{\nu+1}(x)}{K_\nu(x)} J_0(rx)\,dx = \frac{4}{\pi^2}\int_0^\infty \frac{K_0(ry)}{Y_\nu^2(y) + J_\nu^2(y)} \frac{dy}{y} - \begin{cases} 2\nu(\gamma + \log \tfrac{1}{2}r) \\ 1 \\ 0 \end{cases}, \quad . \; . \; . \quad \text{(xi)}$$

according as ν is positive, zero or negative, In other words (ix) is valid for $t = 1$, a result which justifies the derivation of (61), and shows that except for a constant of integration the values of V_s as derived in Sections 6 and 8 are equivalent while the values of $\bar{\rho}/\rho_s$ are identical.

[The writer is greatly indebted to the referees for valuable criticism and advice on several points involving mathematical rigour.]

* WATSON, § 13.21 (8), § 13.3 (8).